

# On linear equations arising in Combinatorics (Part III)

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## 1 Introduction

In the first two papers [1, 2] the author embarked on a study of classes of linear equations over integers satisfying a "Farkas-type" property. As the third paper in this study, the present paper deals with another class of linear equations over integers that has a similar "Farkas-type" property. Furthermore it is shown that if an arbitrary system of equations over integers satisfies the conditions imposed by Farkas' lemma then it has rational solutions of a special type (Theorem 3.3).

## 2 Class $\mathcal{E}_n$

In the first paper [1], it is shown that if a system of linear equations has a suitable property then the existence of an integral solution is decided by a certain set of inequalities (Theorem 3.1 in [1]). In this part, a similar result is presented for another class of linear equations over integers.

### 2.1 Preliminaries

Let  $v = (v_1, \dots, v_n) \in \mathbb{Q}^n$  and let  $s$  be the number of nonzero components of  $v$ . We want to define a linear map  $L_v : \mathbb{Q}^n \rightarrow \mathbb{Q}^{\binom{s}{2}+n-s}$  depending on  $v$ . To present a notationally simpler definition, we assume that  $v_{s+1} = \dots = v_n = 0$ . The linear map  $L_v : \mathbb{Q}^n \rightarrow \mathbb{Q}^{\binom{s}{2}+n-s}$  is defined by the following rule

$$L_v(t_1, \dots, t_n) = \left( \frac{t_1}{v_1} - \frac{t_2}{v_2}, \dots, \frac{t_i}{v_i} - \frac{t_j}{v_j}, \dots, \frac{t_{s-1}}{v_{s-1}} - \frac{t_s}{v_s}, t_{s+1}, \dots, t_n \right) \quad (2.1)$$

Given two elements  $v = (v_1, \dots, v_n), w = (w_1, \dots, w_n) \in \mathbb{Z}^n$ , we write  $v|w$  if  $w_i$  is divisible by  $v_i$  whenever  $v_i \neq 0$ . Let  $A$  be an abelian subgroup of  $\mathbb{Z}^n$  and let  $v \in A$  be an element of  $A$  such that  $v|w$  for all  $w \in A$ . Then it is easy to see that  $L_v(A) \subset \mathbb{Z}^{\binom{s}{2}+n-s}$  is a subgroup of  $\mathbb{Z}^{\binom{s}{2}+n-s}$  and  $L_v : A \rightarrow \mathbb{Z}^{\binom{s}{2}+n-s}$

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defines a  $\mathbb{Z}$ -linear map. Furthermore the kernel of  $L_v$  is  $\mathbb{Z}v$  which in particular implies that the rank of  $L_v(A)$  (as an abelian group) is equal to  $(\text{rank of } A) - 1$ .

We inductively define the notion of a mod-linear function  $l : \mathbb{Z}^n \rightarrow \mathbb{Z}$  of order  $\leq r$  where  $r$  is a nonnegative integer. A mod-linear function of order  $\leq 0$  is just one of the projection maps  $P_i : \mathbb{Z}^n \rightarrow \mathbb{Z}$ ,  $P_i(x_1, \dots, x_n) = x_i$ . When  $r > 0$ , a function  $l : \mathbb{Z}^n \rightarrow \mathbb{Z}$  is called a mod-linear function of order  $\leq r$  if there exist mod-linear functions  $l_1, l_2 : \mathbb{Z}^n \rightarrow \mathbb{Z}$  of order  $\leq r - 1$  and nonzero integers  $m_1, m_2$  such that

$$l(x_1, \dots, x_n) = \lfloor \frac{l_1(x_1, \dots, x_n)}{m_1} \rfloor - \lceil \frac{l_2(x_1, \dots, x_n)}{m_2} \rceil$$

for every  $(x_1, \dots, x_n) \in \mathbb{Z}$ . Here, the notations  $\lfloor x \rfloor$  and  $\lceil x \rceil$  denote the floor and ceiling functions respectively.

Finally we define inductively a subset  $\mathcal{E}_n$  of the set of abelian subgroups of  $\mathbb{Z}^n$  as follows. A nonzero abelian group  $A \subset \mathbb{Z}^n$  belongs to  $\mathcal{E}_n$  if and only if there exists a nonzero vector  $v = (v_1, \dots, v_n) \in A$  satisfying the following properties: (1)  $v|w$  for all  $w \in A$ , and (2)  $L_v(A) = \{0\}$  or  $L_v(A) \in \mathcal{E}_{(s)+n-s}$  where  $L_v$  is defined via 2.1.

## 2.2 A Farkas-type result for $\mathcal{E}_n$

The following theorem can be considered as a generalization of Theorem 3.1 in [1].

**Theorem 2.1.** *For every  $A \in \mathcal{E}_n$  of rank  $r$ , there exists a finite set  $E$ , consisting of mod-linear functions  $l : \mathbb{Z}^{2n} \rightarrow \mathbb{Z}$  of order  $\leq r$ , for which the following statement holds: For arbitrary integers  $a_1 \leq b_1, \dots, a_n \leq b_n$ , there exists an element  $(x_1, \dots, x_n) \in A$  such that  $a_1 \leq x_1 \leq b_1, \dots, a_n \leq x_n \leq b_n$  if and only if for every  $l \in E$  we have  $0 \leq l(a_1, b_1, \dots, a_n, b_n)$ .*

*Proof.* This is proved by induction on  $r$ . First suppose  $r = 1$ . Then there exists an element  $v = (v_1, \dots, v_n) \in A$  such that  $A = \mathbb{Z}v$ . Without loss of generality, we may assume that  $v_1, \dots, v_q > 0$ ,  $v_{q+1}, \dots, v_s < 0$  and  $v_{s+1} = \dots = v_n = 0$ . It is obvious that there exists an element  $(x_1, \dots, x_n) \in A$  such that  $a_1 \leq x_1 \leq b_1, \dots, a_n \leq x_n \leq b_n$ , if and only if there exists an integer  $t$  such that  $a_1 \leq tv_1 \leq b_1, \dots, a_n \leq tv_n \leq b_n$ , or equivalently

$$\begin{aligned} \frac{a_1}{v_1} \leq t \leq \frac{b_1}{v_1}, \dots, \frac{a_q}{v_q} \leq t \leq \frac{b_q}{v_q}, \\ \frac{b_{q+1}}{v_{q+1}} \leq t \leq \frac{a_{q+1}}{v_{q+1}}, \dots, \frac{b_s}{v_s} \leq t \leq \frac{a_s}{v_s}, \\ a_{s+1} \leq 0 \leq b_{s+1}, \dots, a_n \leq 0 \leq b_n. \end{aligned}$$

It is easy to see that these inequalities have a common solution  $t \in \mathbb{Z}$  if and only if the following conditions hold

$$\begin{aligned}
0 &\leq \lfloor \frac{b_j}{v_j} \rfloor - \lceil \frac{a_i}{v_i} \rceil \quad \text{when } 1 \leq i, j \leq q, \\
0 &\leq \lfloor \frac{a_j}{v_j} \rfloor - \lceil \frac{b_i}{v_i} \rceil \quad \text{when } q < i, j \leq s, \\
0 &\leq \lfloor \frac{a_j}{v_j} \rfloor - \lceil \frac{a_i}{v_i} \rceil \quad \text{when } 1 \leq i \leq q < j \leq s, \\
0 &\leq \lfloor \frac{b_i}{v_i} \rfloor - \lceil \frac{b_j}{v_j} \rceil \quad \text{when } 1 \leq i \leq q < j \leq s, \\
0 &\leq b_{s+1} - a_{s+1}, \dots, 0 \leq b_n - a_n.
\end{aligned}$$

Using these inequalities, one can easily construct a desired set  $E$  for  $A$ .

Now suppose  $r > 1$ . Since  $A \in \mathcal{E}_n$ , there exists a nonzero element  $v = (v_1, \dots, v_n) \in A$  satisfying the following properties: (1)  $v|w$  for every  $w \in A$ , and (2)  $L_v(A) \in \mathcal{E}_{\binom{s}{2}+n-s}$ , where  $s$  is the number of nonzero components of  $v$ . I claim that there exists a subgroup  $B$  of  $A$  such that  $A = B \oplus \mathbb{Z}v$ . It is known that such a subgroup  $B$  exists if and only if the abelian group  $A/\mathbb{Z}v$  is torsion-free, i.e. if  $mw \in \mathbb{Z}v$  for a nonzero element  $w \in A$  and a nonzero integer  $m$  then  $w \in \mathbb{Z}v$ . Suppose such an element  $w = (w_1, \dots, w_n)$  and an integer  $m$  exist. There is nothing to prove if  $w = 0$ . So let  $w \neq 0$ . There exists a nonzero integer  $b$  such that  $mw = bv$ . Since  $m \neq 0$ , we see that  $w_i = 0$  if and only if  $v_i = 0$  for all  $i = 1, \dots, n$ . From  $v|w$ , it follows that for all  $i$  such that  $v_i \neq 0$ , we have  $w_i = b_i v_i$  where  $b_i$  is an integer. Therefore we have  $mb_i v_i = mw_i = bv_i$ , implying  $mb_i = b$ . It follows that  $b$  is divisible by  $m$  and consequently  $w = \frac{b}{m}v \in \mathbb{Z}v$ . The proof of the claim is complete.

Without loss of generality, we may assume that  $v_1, \dots, v_q > 0$ ,  $v_{q+1}, \dots, v_s < 0$  and  $v_{s+1} = \dots = v_n = 0$ . Every  $x \in A$  can be written as  $x = tv + y$  where  $t \in \mathbb{Z}$  and  $y \in B$ . It follows that there exists an element  $x = (x_1, \dots, x_n) \in A$  such that  $a_1 \leq x_1 \leq b_1, \dots, a_n \leq x_n \leq b_n$ , if and only if there exist an integer  $t$  and an element  $y = (y_1, \dots, y_n) \in B$  such that  $a_1 \leq tv_1 + y_1 \leq b_1, \dots, a_n \leq tv_n + y_n \leq b_n$ , or equivalently

$$\begin{aligned}
\frac{a_1 - y_1}{v_1} &\leq t \leq \frac{b_1 - y_1}{v_1}, \dots, \frac{a_q - y_q}{v_q} \leq t \leq \frac{b_q - y_q}{v_q}, \\
\frac{b_{q+1} - y_{q+1}}{v_{q+1}} &\leq t \leq \frac{a_{q+1} - y_{q+1}}{v_{q+1}}, \dots, \frac{b_s - y_s}{v_s} \leq t \leq \frac{a_s - y_s}{v_s}, \\
a_{s+1} &\leq y_{s+1} \leq b_{s+1}, \dots, a_n \leq y_n \leq b_n.
\end{aligned}$$

One can easily see that these equations have a common solution  $t \in \mathbb{Z}$  if and only if the following inequalities hold

$$0 \leq \lfloor \frac{b_j - y_j}{v_j} \rfloor - \lceil \frac{a_i - y_i}{v_i} \rceil \quad \text{when } 1 \leq i, j \leq q,$$

$$\begin{aligned}
0 &\leq \lfloor \frac{a_j - y_j}{v_j} \rfloor - \lceil \frac{b_i - y_i}{v_i} \rceil \quad \text{when } q < i, j \leq s, \\
0 &\leq \lfloor \frac{a_j - y_j}{v_j} \rfloor - \lceil \frac{a_i - y_i}{v_i} \rceil \quad \text{when } 1 \leq i \leq q < j \leq s, \\
0 &\leq \lfloor \frac{b_i - y_i}{v_i} \rfloor - \lceil \frac{b_j - y_j}{v_j} \rceil \quad \text{when } 1 \leq i \leq q < j \leq s, \\
a_{s+1} &\leq y_{s+1} \leq b_{s+1}, \dots, a_n \leq y_n \leq b_n.
\end{aligned}$$

Using the fact that  $v|w$  for all  $w \in A$ , one can show that these conditions are equivalent to the following conditions:

(1) For all  $1 \leq i < j \leq q$ , we have

$$\lceil \frac{a_i}{v_i} \rceil - \lfloor \frac{b_j}{v_j} \rfloor \leq \frac{y_i}{v_i} - \frac{y_j}{v_j} \leq \lfloor \frac{b_i}{v_i} \rfloor - \lceil \frac{a_j}{v_j} \rceil$$

(1') For all  $1 \leq i \leq q$ , we have

$$0 \leq \lfloor \frac{b_i}{v_i} \rfloor - \lceil \frac{a_i}{v_i} \rceil$$

(2) For all  $q < i < j \leq s$ , we have

$$\lceil \frac{b_i}{v_i} \rceil - \lfloor \frac{a_j}{v_j} \rfloor \leq \frac{y_i}{v_i} - \frac{y_j}{v_j} \leq \lfloor \frac{a_i}{v_i} \rfloor - \lceil \frac{b_j}{v_j} \rceil$$

(2') For all  $q < j \leq s$ , we have

$$0 \leq \lfloor \frac{a_j}{v_j} \rfloor - \lceil \frac{b_j}{v_j} \rceil$$

(3) For all  $1 \leq i \leq q < j \leq s$ , we have

$$\lceil \frac{a_i}{v_i} \rceil - \lfloor \frac{a_j}{v_j} \rfloor \leq \frac{y_i}{v_i} - \frac{y_j}{v_j} \leq \lfloor \frac{b_i}{v_i} \rfloor - \lceil \frac{b_j}{v_j} \rceil$$

(4)

$$a_{s+1} \leq y_{s+1} \leq b_{s+1}, \dots, a_n \leq y_n \leq b_n.$$

Put

$$a_{ij} = \begin{cases} \lceil \frac{a_i}{v_i} \rceil - \lfloor \frac{b_i}{v_j} \rfloor & \text{when } 1 \leq i < j \leq q \\ \lceil \frac{b_i}{v_i} \rceil - \lfloor \frac{a_j}{v_j} \rfloor & \text{when } q < i < j \leq s \\ \lceil \frac{a_i}{v_i} \rceil - \lfloor \frac{a_j}{v_j} \rfloor & \text{when } 1 \leq i \leq q < j \leq s \end{cases}$$

and

$$b_{ij} = \begin{cases} \lfloor \frac{b_i}{v_i} \rfloor - \lceil \frac{a_j}{v_j} \rceil & \text{when } 1 \leq i < j \leq q \\ \lfloor \frac{a_i}{v_i} \rfloor - \lceil \frac{b_j}{v_j} \rceil & \text{when } q < i < j \leq s \\ \lfloor \frac{b_i}{v_i} \rfloor - \lceil \frac{b_j}{v_j} \rceil & \text{when } 1 \leq i \leq q < j \leq s. \end{cases}$$

To complete the proof we need the following lemma.

**Lemma 2.2.** *There exists an element  $x = (x_1, \dots, x_n) \in A$  such that  $a_1 \leq x_1 \leq b_1, \dots, a_n \leq x_n \leq b_n$  if and only if for all  $1 \leq i \leq q$  we have  $0 \leq \lfloor \frac{b_i}{v_i} \rfloor - \lceil \frac{a_i}{v_i} \rceil$ , for all  $q < j \leq s$  we have  $0 \leq \lfloor \frac{a_j}{v_j} \rfloor - \lceil \frac{b_j}{v_j} \rceil$ , and there exists an element  $(z_{12}, \dots, z_{(s-1)s}, z_1, \dots, z_{n-s}) \in L_v(A)$  such that  $a_{ij} \leq z_{ij} \leq b_{ij}$  for all  $1 \leq i < j \leq s$  and  $a_{s+1} \leq z_1 \leq b_{s+1}, \dots, a_n \leq z_{n-s} \leq b_n$ .*

*Proof.* First suppose there exists an element  $x = (x_1, \dots, x_n) \in A$  such that  $a_1 \leq x_1 \leq b_1, \dots, a_n \leq x_n \leq b_n$ . Since  $A = B \oplus \mathbb{Z}v$ , there exist an integer  $t$  and an element  $(y_1, \dots, y_n) \in B$  such that  $x_1 = y_1 + tv_1, \dots, x_n = y_n + tv_n$ . As shown above, it follows that  $y_1, \dots, y_n$  satisfy Inequalities (1), (1'), (2), (2'), (3), and (4) above, which in particular implies that the vector  $L_v(x) \in L_v(A)$  satisfies the desired conditions in the lemma and we are done.

Conversely, suppose there exists an element  $(z_{12}, \dots, z_{(s-1)s}, z_1, \dots, z_{n-s}) \in L_v(A)$  such that  $a_{ij} \leq z_{ij} \leq b_{ij}$  for all  $1 \leq i < j \leq s$  and  $a_{s+1} \leq z_1 \leq b_{s+1}, \dots, a_n \leq z_{n-s} \leq b_n$ . Since  $L_v(B) = L_v(A)$ , there exists an element  $(y_1, \dots, y_n) \in B$  such that  $L_v(y_1, \dots, y_n) = (z_{12}, \dots, z_{(s-1)s}, z_1, \dots, z_{n-s})$ . It follows that  $y_1, \dots, y_n$  satisfy Inequalities (1), (2), (3), and (4) above. Furthermore, by assumption, Inequalities (1') and (2') hold. As shown above, it follows that there exists an element  $(x_1, \dots, x_n) \in A$  such that  $a_1 \leq x_1 \leq b_1, \dots, a_n \leq x_n \leq b_n$ .  $\square$

The group  $L_v(A)$  has a smaller rank than the group  $A$  and  $L_v(A) \in \mathcal{E}_{\binom{s}{2}+n-s}$ . By induction, there exists a finite set  $E'$  of mod-linear functions for  $A$  which satisfy the corresponding conditions. Since  $(-a_{ij})$ 's and  $b_{ij}$ 's are mod-linear functions (of order  $\leq 1$ ) in terms of  $a_i$ 's and  $b_j$ 's one can easily show that each element of  $E'$  gives rise to a mod-linear function (of order  $\leq r$ ) in terms of  $a_i$ 's and  $b_j$ 's. Let  $L(E')$  be the set of such mod-linear functions in terms of  $a_i$ 's and  $b_j$ 's. Inequalities (1') and (2') give rise to a finite set  $E''$  consisting of mod-linear functions of order  $\leq 1$ . Using Lemma 2.2, one can easily see that the set  $E = L(E') \cup E''$  satisfies the desired condition in the lemma and therefore the proof is complete.  $\square$

### 3 Rational solutions of special types

In this part, it is shown that if a system of linear equations over integers has a rational solution in some interval then it has rational solutions of a particular type in the same interval.

**Definition 3.1.** *Let  $v_1, \dots, v_m \in \mathbb{Q}^n$  be arbitrary vectors. Depending on  $v_1, \dots, v_m$ , the set  $P_{v_1, \dots, v_m}$  is defined to be the set of primes  $p$  for which there exists an integral elementary relation  $\sum_{i=1}^m a_i v_i = 0$  such that  $p \mid \prod_{a_i \neq 0} a_i$ .*

It is known that there exist only finitely many elementary integral relations among  $v_1, \dots, v_m$  (see [3]). This implies that the set  $P_{v_1, \dots, v_m}$  is a finite (possibly

empty) set. Given a set of primes  $P$ , let  $\mathbb{Q}_P$  denote the following ring

$$\mathbb{Q}_P = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \text{ and all prime factors of } b \text{ belong to } P \right\}.$$

**Lemma 3.1.** *Let  $v_1, \dots, v_m \in \mathbb{Q}^n$  be given and put  $P = P_{v_1, \dots, v_m}$ . Let  $w \in \sum_{i=1}^m \mathbb{Q}_P v_i$  and assume that there exists a natural number  $k$ , with no prime factors in  $P$ , and a set  $I \subset \{1, \dots, m\}$  such that  $kw \in \sum_{i \in I} \mathbb{Q}_P v_i$ . Then we have  $w \in \sum_{i \in I} \mathbb{Q}_P v_i$ .*

*Proof.* The proof is by induction on  $m - |I|$ . There is nothing to prove when  $m - |I| = 0$ , so suppose  $m - |I| > 0$ . Without loss of generality, we may assume that  $m \notin I$ . By induction on  $m - |I|$ , we have  $w \in \sum_{i \in I \cup \{m\}} \mathbb{Q}_P v_i$ , i.e.  $w = \sum_{i \in I \cup \{m\}} b_i v_i$ , where  $b_i \in \mathbb{Q}_P$  ( $i \in I \cup \{m\}$ ). If  $b_m = 0$ , then we are done. So suppose  $b_m \neq 0$ . We have  $kb_m v_m \in \sum_{i \in I} \mathbb{Q}_P v_i$ . There exists a nonempty set  $J \subset I$ , such that the vectors  $\{v_j\}_{j \in J}$  are linearly independent and  $kb_m v_m \in \sum_{i \in J} \mathbb{Q}_P v_i$ . It follows that there exists an elementary integral relation  $\sum_{i \in J \cup \{m\}} a_i v_i = 0$ . Since the vectors  $\{v_j\}_{j \in J}$  are linearly independent, we have  $a_m \neq 0$ . Moreover  $k$  and  $a_m$  are relatively prime, by virtue of the assumption on  $k$ . Since  $k(b_m v_m), a_m(b_m v_m) \in \sum_{i \in I} \mathbb{Q}_P v_i$ , we deduce that  $b_m v_m \in \sum_{i \in I} \mathbb{Q}_P v_i$  which implies that  $w \in \sum_{i \in I} \mathbb{Q}_P v_i$  because  $w = \sum_{i \in I \cup \{m\}} b_i v_i$ .  $\square$

**Theorem 3.2.** *Let  $v_1, \dots, v_m \in \mathbb{Q}^n$  and  $a_1 \leq b_1, \dots, a_m \leq b_m$  be in  $\mathbb{Q}_P$  where  $P = P_{v_1, \dots, v_m}$ . If a vector  $w \in \sum_{i=1}^m \mathbb{Q}_P v_i$  can be written as  $w = \sum_{i=1}^m x_i v_i$  where  $a_1 \leq x_1 \leq b_1, \dots, a_m \leq x_m \leq b_m$  are rational numbers, then there exist numbers  $a_1 \leq y_1 \leq b_1, \dots, a_m \leq y_m \leq b_m$  in  $\mathbb{Q}_P$  such that  $w = \sum_{i=1}^m y_i v_i$ .*

*Proof.* In the case  $P = \emptyset$ , the theorem is proved in [1] (Theorem 3.3). In what follows we assume that  $P \neq \emptyset$ . The proof is by induction on  $m$ . First let  $m = 1$ . Since  $w \in \mathbb{Q}_P v_1$ , we have  $w = l v_1$  where  $l \in \mathbb{Q}_P$ , implying that  $w = x_1 v_1 = l v_1$ . If  $v_1 = 0$ , then  $y_1 = a_1$  satisfies the condition. If  $v_1 \neq 0$ , then  $x_1 = l$  and we are done.

Now let  $m > 1$ . If the vectors  $v_1, \dots, v_m$  are linearly independent, then from  $w = \sum_{i=1}^m x_i v_i$  and  $w \in \sum_{i=1}^m \mathbb{Q}_P v_i$ , it follows that  $x_1, \dots, x_m \in \mathbb{Q}_P$  and we are done. So we may assume that  $v_1, \dots, v_m$  are  $\mathbb{Z}$ -linearly dependent. It is easy to see that there exists a natural number  $k$  with no prime factors in  $P = P_{v_1, \dots, v_m}$  such that each  $x_i$  can be written as  $x_i = \frac{N_i}{k}$  where  $N_i \in \mathbb{Q}_P$ . We consider two cases.

*Case 1:* Assume that there exists a coefficient, say  $x_1$ , which belongs to  $\mathbb{Q}_P$ . From  $k(w - x_1 v_1) = \sum_{i=2}^m N_i v_i$  and Lemma 3.1, it follows that  $w - x_1 v_1 \in \sum_{i=2}^m \mathbb{Q}_P v_i$ . Set  $P' = P_{v_2, \dots, v_m}$ . It is clear that  $P' \subset P$  and  $\mathbb{Q}_{P'} \subset \mathbb{Q}_P$  using which one can easily show that there exists a natural number  $M$  whose prime factors belong to  $P \setminus P'$ , such that  $M(w - x_1 v_1) \in \sum_{i=2}^m \mathbb{Q}_{P'} v_i$ . Considering the relation  $M(w - x_1 v_1) = \sum_{i=2}^m (M x_i) v_i$ , we see that by induction there exists numbers  $M a_2 \leq y'_2 \leq M b_2, \dots, M a_m \leq y'_m \leq M b_m$ , all in  $\mathbb{Q}_{P'}$ , such that  $M(w - x_1 v_1) = \sum_{i=2}^m y'_i v_i$ . The presentation  $w = x_1 v_1 + \sum_{i=2}^m \frac{y'_i}{M} v_i$  satisfies the

desired conditions and we are done.

*Case 2:* Assume that none of the coefficients  $x_1, \dots, x_m$  belong to  $\mathbb{Q}_P$ . In particular, we have  $ka_i < N_i < kb_i$  for every  $i = 1, \dots, m$ . Since  $v_1, \dots, v_m$  are linearly dependent, there exists an elementary integral relation  $\sum_{i=1}^m c_i v_i = 0$ . Without loss of generality, we may assume  $c_1 \neq 0$ . One can easily prove that  $\mathbb{Q}_P$  is dense in  $\mathbb{R}$  when  $P \neq \emptyset$ . Since  $\mathbb{Q}_P$  is dense in  $\mathbb{R}$  and  $ka_i < N_i < kb_i$  for all  $i = 1, \dots, m$ , one is able to find a rational number  $r$  such that  $ka_1 \leq N_1 + rc_1 = ky_1 \leq kb_1$ , where  $y_1 \in \mathbb{Q}_P$  and  $ka_i \leq N_i + rc_i \leq kb_i$  for all  $i = 2, \dots, m$ . Note that since  $c_1$  is invertible in  $\mathbb{Q}_P$ , we have  $r \in \mathbb{Q}_P$  which implies that  $N_i + rc_i \in \mathbb{Q}_P$  for all  $i = 1, 2, \dots, m$ . Now we have  $w = \sum_{i=1}^m \frac{N_i + rc_i}{k} v_i$  where  $\frac{N_i + rc_i}{k} \in \mathbb{Q}_P$  and  $ka_i \leq N_i + rc_i \leq kb_i$  for all  $i = 1, 2, \dots, m$ . We can now use Case 1 to complete the proof.  $\square$

Using Farkas' lemma over  $\mathbb{Q}$  (Theorem 2.4 in [1]), one can easily derive the following result.

**Theorem 3.3.** *Let  $v_1, \dots, v_m \in \mathbb{Q}^n$  and  $a_1 \leq b_1, \dots, a_m \leq b_m$  be in  $\mathbb{Q}_P$  where  $P = P_{v_1, \dots, v_m}$ . Then a vector  $w \in \mathbb{Q}^n$  can be written as  $w = \sum_{i=1}^m x_i v_i$  where  $a_1 \leq x_1 \leq b_1, \dots, a_m \leq x_m \leq b_m$  belong to  $\mathbb{Q}_P$  if and only if  $w \in \sum_{i=1}^m \mathbb{Q}_P v_i$  and*

$$(u, w) \leq \sum_{i=1}^m a_i \frac{(u, v_i) - |(u, v_i)|}{2} + \sum_{i=1}^m b_i \frac{(u, v_i) + |(u, v_i)|}{2},$$

for every  $\{v_1, \dots, v_m\}$ -indecomposable point  $[u] \in \mathbb{RP}_+^{n-1}$ .

## References

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